JOURNAL OF APPROXIMATION THEORY **85**, 250–268 (1996)
ARTICLE NO. 0041

Best Interpolatory Approximation in Normed Linear Spaces

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Communicated by Günther Nürnberger

Received December 22, 1994; accepted June 6, 1995

A theory of best approximation with interpolatory contraints from a finite-dimensional subspace M of a normed linear space X is developed. In particular, to each $x \in X$, best approximations are sought from a subset $M(x)$ of M which **depends** on the element x being approximated. It is shown that this "parametric approximation" problem can be essentially reduced to the "usual" one involving a certain **fixed** subspace M_0 of M. More detailed results can be obtained when (1) X is a Hilbert space, or (2) M is an "interpolating subspace" of X (in the sense of [1]).

1. INTRODUCTION

In this section we establish the notation and terminology that is used throughout. In Section 2 we describe the problem of approximation with interpolatory constraints from a finite-dimensional subspace M of the normed linear space X. By using a perturbation technique, we show that the problem of parametric approximation with interpolatory constraints can be reduced to ordinary best approximation from a **fixed** subspace of M. A general theory of best parametric approximation is developed which includes existence and characterization theorems, as well as continuity criteria for the (set-valued) parameter mapping and selection properties of this mapping.

In Section 3 we specialize X to Hilbert space and deduce some stronger results. In Section 4 we restrict our attention to interpolating subspaces [1]. In this case, there is a substantial strengthening of the theory that can be obtained. In particular, best interpolatory approximations are always

* This author was supported in part by NSF Grant DMS-930375.

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0021-9045/96 $18.00

(strongly) unique, and the parameter mapping is pointwise Lipschitz continuous. We should note that in the particular case when $X = C[a, b]$ and M is a Haar subspace of $C[a, b]$, this problem had been considered earlier by the first author [4]. However, even specialized to this particular situation, some of the results of the present paper are stronger and more general than those of [4]. The results of [4] were also extended to "ε-interpolation" by Mabizela and Zhong [9].

Let K be a closed convex subset of a normed linear space X. For a given $x \in X$, the (possibly empty) set of all **best approximations** to x from K is defined by

$$P_K(x) := \{ y \in K \mid \|x - y\| = d(x, K) \},$$

where $d(x, K) := \inf\{ \|x - y\| \mid y \in K \}$. K is said to be **proximinal** (resp., **Chebyshev**) if for each $x \in X$, the set $P_K(x)$ is nonempty (resp., a singleton).

Unless otherwise stated, X will always denote a (real) normed linear space and X^* the dual space of all continuous linear functionals on X.

2. The Interpolation Problem

Let M be an n-dimensional subspace of a normed linear space X and $\{\phi_1, \phi_2, ..., \phi_m\} \subset X^*$ be a set of $m \leqslant n$ linearly independent functionals. For each $x \in X$, let

$$M(x) := \{ y \in M \mid \phi_i(y) = \phi_i(x), i = 1, 2, ..., m \}.$$

Since $M(x)$ is not changed if the ϕ_i are scaled, we may (and will) assume that $\|\phi_i\| = 1$ for each i. The elements of $M(x)$ are said to **interpolate** x relative to the set $\{\phi_1, \phi_2, ..., \phi_m\}$. Thus an element $x_0 \in M(x)$ is a best approximation to x from $M(x)$ provided that

$$\|x - x_0\| = d(x, M(x)),$$

and the set of best approximations to x from $M(x)$ is $P_{M(x)}(x)$. Note that, unlike the standard case of approximating from a fixed set, the set $M(x)$ that one approximates from *depends* on the point x being approximated. Such problems are often called parametric approximation problems and $P_{M(\cdot)}(\cdot)$ is called the **parameter map**. Of course, if $x \in M$, then $x \in M(x)$ and so x is its own best approximation from $M(x)$: $P_{M(x)}(x) = \{x\}$.

For m elements $y_1, y_2, ..., y_m$ in X, we define the determinant

$$\det[\phi_i(y_j)]_{1 \leqslant i, j \leqslant m} := \begin{vmatrix} \phi_1(y_1) & \phi_1(y_2) & \cdots & \phi_1(y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_m(y_1) & \phi_m(y_2) & \cdots & \phi_m(y_m) \end{vmatrix}.$$

Our first result establishes useful conditions each of which is equivalent to the statement that $M(x) \neq \emptyset$ for each $x \in X$.

LEMMA 2.1. *The following statements are equivalent.*

(1) *For each* $x \in X$, $M(x) \neq \emptyset$;

(2) *The set* $\{\phi_1, \phi_2, ..., \phi_m\}$ *is linearly independent over* M (*i.e.*, $\sum_1^m \alpha_i \phi_i(y) = 0$ *for all* $y \in M$ *implies* $\alpha_i = 0$ *for all* i);

(3) *There exist elements* $z_1, z_2, ..., z_m$ *in* M *such that*

$$\phi_i(z_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j; \end{cases}$$

(4) *There exist elements* $z_1, z_2, ..., z_m$ *in* M *such that*

$$\det[\phi_i(z_j)] \neq 0.$$

Proof. (1) $\Rightarrow$ (2). Assume $M(x) \neq \emptyset$ for all $x \in X$. If $\sum_1^m \alpha_i \phi_i(y) = 0$ for all $y \in M$, then $\sum_1^m \alpha_i \phi_i(x) = 0$ for all $x \in X$. By linear independence of $\{\phi_1, \phi_2, ..., \phi_m\}$, $\alpha_i = 0$ for all i. Thus (2) holds.

(2) $\Rightarrow$ (3). We proceed by induction on m. For $m = 1$, there exists $y \in M$, so that $\phi_1(y) \neq 0$. Then $y_1 := y/\phi_1(y) \in M$ satisfies $\phi_1(y_1) = 1$. Now suppose (3) is valid for $m = k$ and $\{\phi_1|_M, \phi_2|_M, ..., \phi_{k+1}|_M\}$ is linearly independent. By hypothesis, there exists $\{y_1, y_2, ..., y_k\}$ in M so that $\phi_i(y_j) = \delta_{ij}$ $(i, j = 1, 2, ..., k)$. We claim that there exists $y \in M$ so that $\phi_i(y) = 0$ for $i = 1, 2, ..., k$, and $\phi_{k+1}(y) \neq 0$. Otherwise, by [8; p. 421], $\phi_{k+1}|_M$ would be a linear combination of $\phi_1|_M, ..., \phi_k|_M$, which contradicts the linear independence of the $\phi_i|_M$'s. Setting $z_{k+1} = y/\phi_{k+1}(y)$ and $z_i = y_i - (\phi_{k+1}(y_i))/(\phi_{k+1}(y))\, y$ $(i = 1, 2, ..., k)$, we see that $z_i \in M$ for all $i = 1, 2, ..., k+1$, and

$$\phi_i(z_j) = \delta_{ij} \qquad (i, j = 1, 2, ..., k+1).$$

This completes the induction.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (1). If (4) holds and $x \in X$, the system of equations

$$\sum_{j=1}^{m} \alpha_j \phi_i(z_j) = \phi_i(x) \qquad (i = 1, 2, ..., m)$$

has a (unique) solution $\alpha_1, ..., \alpha_m$. Then the element $y = \sum_{j=1}^m \alpha_j z_j$ is in M and $\phi_i(y) = \phi_i(x)$ $(i = 1, 2, ..., m)$. That is, $y \in M(x)$. This proves (1). ∎

Let

$$X_0 := \{x \in X \mid \phi_i(x) = 0 \qquad (i = 1, 2, ..., m)\} \tag{2.1.1}$$

and

$$M_0 := M(0) = \{y \in M \mid \phi_i(y) = 0 \qquad (i = 1, 2, ..., m)\} = M \cap X_0. \tag{2.1.2}$$

LEMMA 2.2. *M_0 is an $(n-m)$-dimensional subspace. In particular, $M_0 = \{0\}$ if $m = n$.*

Proof. Since M is n-dimensional, so is its dual M^*. Since $\{\phi_1|_M, ..., \phi_m|_M\}$ is linearly independent in M^*, there are functionals $\{\psi_{m+1}, ..., \psi_n\}$ in M^* so that $\{\phi_1|_M, ..., \phi_m|_M, \psi_{m+1}, ..., \psi_n\}$ is linearly independent. By the Hahn-Banach theorem, each ψ_i can be extended to a $\phi_i \in X^*$. Thus $\{\phi_1|_M, ..., \phi_m|_M, \phi_{m+1}|_M, ..., \phi_n|_M\}$ is linearly independent. By Lemma 2.1, there exists a set $\{y_1, y_2, ..., y_n\}$ in M so that

$$\phi_i(y_j) = \delta_{ij} \qquad (i, j = 1, 2, ..., n). \tag{2.2.1}$$

$\{y_1, y_2, ..., y_n\}$ is clearly linearly independent and hence a basis for M. Since $\{y_{m+1}, ..., y_n\}$ is in M_0 by (2.2.1), it follows that $\dim M_0 \geqslant n - m$. On the other hand, for each $y \in M_0$, y is in M so $y = \sum_1^n \alpha_i y_i$ for some scalars α_i. We have for each $j \leqslant m$,

$$0 = \phi_j(y) = \sum_{i=1}^{m} \alpha_i \phi_j(y_i) = \alpha_j.$$

Thus $y = \sum_{m+1}^n \alpha_i y_i$ so that $M_0 \subset \operatorname{span}\{y_{m+1}, ..., y_n\}$ and hence $\dim M_0 \leqslant n - m$. This proves that $\dim M_0 = n - m$. ▌

To avoid vacuous or trivial statements, we shall assume hereafter in this section that $M(x) \neq \varnothing$ for each $x \in X$. Thus (by Lemma 2.1) there exists a linearly independent set $\{z_1, z_2, ..., z_m\}$ in M such that

$$\phi_i(z_j) = \delta_{ij} \qquad (i, j = 1, 2, ..., m). \tag{2.2.2}$$

Fixing such a set $\{z_1, z_2, ..., z_m\}$, we define an operator $L\colon X \to M$ by

$$Lx = \sum_1^m \phi_i(x)\, z_i, \qquad x \in X. \tag{2.2.3}$$

It turns out that L is a (linear) *projection* onto the subspace $\operatorname{span}\{z_1, z_2, ..., z_m\}$.

LEMMA 2.3. (1) *L is a bounded linear operator.*

(2) *For each $z \in \operatorname{span}\{z_1, z_2, ..., z_m\}$, $Lz = z$.*

(3) *L is "idempotent", i.e., $L^2x = Lx$ for all x.*

(4) $X_0 = \{x - Lx \mid x \in X\} = L^{-1}(0)$.

(5) $X = X_0 \oplus \operatorname{span}\{z_1, z_2, ..., z_m\}$.

Proof. (1) This is clear.

(2) By linearity, it suffices to show that $Lz_j = z_j$ for each $j = 1, 2, ..., m$. But

$$Lz_j = \sum_{i=1}^{m} \phi_i(z_j)\, z_i = z_j \qquad (j = 1, 2, ..., m).$$

(3) For each $x \in X$, using (2), we have

$$L^2x = L(Lx) = L\left(\sum_{i=1}^{m} \phi_i(x)\, z_i\right) = \sum_1^m \phi_i(x)\, Lz_i = \sum_1^m \phi_i(x)\, z_i = Lx.$$

(4) Clearly, $x \in X_0$ iff $\phi_i(x) = 0$ for all i iff $\sum_1^m \phi_i(x)\, z_i = 0$ (since $\{z_1, z_2, ..., z_m\}$ is linearly independent) iff $Lx = 0$ iff $x \in L^{-1}(0)$. Thus $X_0 = L^{-1}(0)$. Let $S = \{x - Lx \mid x \in X\}$. If $x \in S$, then $x = y - Ly$ for some y implies $Lx = Ly - L^2y = 0$ by (3). Thus $S \subset L^{-1}(0)$. Conversely, if $x \in L^{-1}(0)$, then $x = x - Lx \in S$. That is, $L^{-1}(0) \subset S$ and thus $L^{-1}(0) = S$.

(5) For each $x \in X$,

$$x = (x - Lx) + Lx \in X_0 + \operatorname{span}\{z_1, z_2, ..., z_m\}.$$

If $x \in X_0 \cap \operatorname{span}\{z_1, z_2, ..., z_m\}$, then $Lx = 0$ and $Lx = x$ (by (1)). Thus $x = 0$ so that $X_0 \cap \operatorname{span}\{z_1, z_2, ..., z_m\} = \{0\}$. This proves (5). ∎

THEOREM 2.4. (1) *For each $x \in X$,*

$$M(x) = M_0 + Lx, \tag{2.4.1}$$

and

$$P_{M(x)}(x) = P_{M_0}(x - Lx) + Lx. \tag{2.4.2}$$

(2) *If $x \in X_0$, then*

$$P_{M(x)}(x) = P_{M_0}(x). \tag{2.4.3}$$

(3) *If $m = n$, then $M(x) = \{Lx\}$ and*

$$P_{M(x)}(x) = Lx. \tag{2.4.4}$$

Proof. (1) Clearly, $Lx \in M$ and $\phi_j(Lx) = \sum_1^m \phi_i(x)\, \phi_j(z_i) = \phi_j(x)$ $(j = 1, 2, ..., m)$ by (2.2.2). Thus $Lx \in M(x)$ and hence $x - Lx \in X_0$. Moreover, $y \in M(x)$ iff $y \in M$ and $\phi_i(y) = \phi_i(x) = \phi_i(Lx)$ for all i iff $y \in M$ and $y - Lx \in X_0 \cap M = M_0$ iff $y \in M_0 + Lx$. This proves (2.4.1).

From (2.4.1), we obtain

$$P_{M(x)}(x) = P_{M_0 + Lx}(x) = P_{M_0}(x - Lx) + Lx,$$

which verifies (2.4.2).

(2) If $x \in X_0$, then $Lx = 0$ by Lemma 2.3(4) and (2.4.3) follows from (2.4.2).

(3) If $m = n$, $M_0 = \{0\}$ by Lemma 2.1 so that $M(x) = \{Lx\}$ by part (1). ∎

The main consequence of Theorem 2.4 is that it shows that *the problem of parametric approximation with interpolatory contraints can be reduced to an ordinary best approximation problem from the* **fixed** *subspace M_0 of M.* As we shall see, it also suggests the study of best approximation of the elements of the subset X_0 by elements of the subspace M_0.

COROLLARY 2.5. (1) *$P_{M(x)}(x) \neq \varnothing$ for each $x \in X$.*

(2) *$P_{M(x)}(x)$ is a singleton for each $x \in X$ if and only if M_0 is a Chebyshev subspace of X_0.*

(3) *If X is strictly convex, then $P_{M(x)}(x)$ is a singleton for each $x \in X$.*

Proof. (1) This follows from (2.4.2) and the fact that every finite-dimensional subspace is proximinal.

(2) This is a consequence of Theorem 2.4(1).

(3) This follows from (2) and the fact that all finite-dimensional subspaces of strictly convex spaces are Chebyshev. ∎

Using Theorem 2.4, a characterization of best approximations to x from $M(x)$ can be obtained by reducing it to a "standard" problem of approximating from a finite-dimensional subspace.

For any set S, $co(S)$ will denote its convex hull: the intersection of all convex sets which contain S. The unit ball in X^* is denoted by $B(X^*)$, the set of extreme points in $B(X^*)$ by ext $B(X^*)$, and the set of "extreme peaking functionals" for $x \in X$ is defined by

$$\mathscr{E}(x) := \{x^* \in \text{ext } B(X^*) \mid x^*(x) = \|x\|\}.$$

THEOREM 2.6. *Let $x \in X$ and $y \in M(x)$. Then the following statements are equivalent.*

(1) $y \in P_{M(x)}(x)$;

(2) $0 \in \operatorname{co}\{(x^*(y_1), x^*(y_2), ..., x^*(y_{n-m})) \mid x^* \in \mathscr{E}(x-y)\}$, *where* $\{y_1, y_2, ..., y_{n-m}\}$ *is any basis for* M_0;

(3) *There exist* k *functionals* $x_i^* \in \mathscr{E}(x-y)$, $1 \leqslant k \leqslant n-m+1$, *and* k *scalars* $\lambda_i > 0$ *such that* $\sum_1^k \lambda_i x_i^* \in M_0^{\perp}$.

Proof. From (2.4.2), we see that $y \in P_{M(x)}(x)$ iff $y - Lx \in P_{M_0}(x - Lx)$. Now M_0 is an $(n-m)$-dimensional subspace by Lemma 2.2. Applying the well-known characterization of best approximations (see [10; Theorem 1.1, p. 170]), we obtain the equivalence of (1) and (3). The equivalence of (2) and (3) is a consequence of Carathéodory's theorem. ∎

Next we show that any continuity property for the set-valued parameter mapping $x \mapsto P_{M(x)}(x)$ is equivalent to the same property for the metric projection onto M_0.

For any two nonempty closed and bounded sets A and B in a metric space Y, define

$$h(A, B) := \sup_{a \in A} d(a, B)$$

and

$$H(A, B) := \max\{h(A, B), h(B, A)\}.$$

In other words, H is the *Hausdorff metric* on the set of all nonempty closed and bounded subsets of Y.

Recall the following continuity concepts for set-valued maps.

DEFINITION 2.7. Let X, Y be metric spaces, $F: X \to 2^Y \setminus \{\varnothing\}$ and $x_0 \in X$. Then F is said to be

(1) **lower Hausdorff semicontinuous** (l.H.s.c.) at x_0 if, for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $h(F(x_0), F(x)) < \varepsilon$ for all $x \in U$;

(2) **upper Hausdorff semicontinuous** (u.H.s.c.) at x_0 if, for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $h(F(x), F(x_0)) < \varepsilon$ for all $x \in U$;

(3) **Hausdorff semicontinuous** (H.s.c.) at x_0 if it is both u.H.s.c. and l.H.s.c. at x_0, i.e., for each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $H(F(x), F(x_0)) < \varepsilon$ for all $x \in U$;

(4) **lower semicontinuous** (l.s.c.) at x_0 if, for any open set V in Y with $F(x_0) \cap V \neq \varnothing$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \varnothing$ for all $x \in U$;

(5) **upper semicontinuous** (u.s.c.) at x_0 if, for any open set V in Y with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$.

It is known (see [7]) that if $F(x_0)$ is compact, then F is u.s.c. (resp., l.s.c.) at x_0 if and only if F is u.H.s.c. (resp., l.H.s.c.) at x_0. Using Theorem 2.4, it is easy to deduce that $P_{M(x)}(x)$ is a closed and bounded, hence compact, subset of M. (In fact, $\|y\| \leqslant (2+3\,\|L\|)\,\|x\|$ for each $y \in P_{M(x)}(x)$.) Using these remarks, we can prove the next result.

THEOREM 2.8. *Let $x_0 \in X$ and $\tau = u, l, u.H$, or $l.H$. Then the following statements are equivalent.*

(1) $P_{M(\cdot)}(\cdot)$ *is τ.s.c. at x_0;*

(2) $P_{M_0} \circ (I-L)$ *is τ.s.c. at x_0;*

(3) $P_{M_0}|_{X_0}$ *is τ.s.c. at $x_0 - Lx_0$.*

Proof. As observed above, $P_{M(x)}(x)$ is compact so l.s.c. = l.H.s.c. (resp., u.s.c. = u.H.s.c.). We will prove the equivalence when $\tau =$ l.H. (= l). The proof when $\tau =$ u.H. (= u) is similar.

(1) ⇒ (2). Let $x \in X$. If $y \in P_{M_0} \circ (I-L)(x_0)$, then $y = z - Lx_0$ for some $z \in P_{M(x_0)}(x_0)$ by (2.4.2). Thus

$$\begin{aligned} d(y, P_M \circ (I-L)(x) &= d(y, P_{M(x)}(x) - Lx) \\ &= d(z - Lx_0, P_{M(x)}(x) - Lx) \\ &\leqslant d(z, P_{M(x)}(x)) + \|Lx - Lx_0\| \\ &\leqslant h(P_{M(x_0)}(x_0), P_{M(x)}(x)) + \|L\|\,\|x - x_0\|. \end{aligned}$$

Thus

$$\begin{aligned} &h(P_{M_0} \circ (I-L)(x_0), P_{M_0} \circ (I-L)(x)) \\ &\quad \leqslant h(P_{M(x_0)}(x_0), P_{M(x)}(x)) + \|L\|\,\|x - x_0\|. \end{aligned}$$

Since $P_{M(\circ)}(\cdot)$ is l.s.c. at x_0, the right side of this inequality can be made arbitrarily close to zero by choosing x sufficiently close to x_0. Thus (2) holds.

(2) ⇒ (3). For any $x \in X$,

$$h(P_{M_0}(x_0 - Lx_0), P_{M_0}(x - Lx)) = h(P_{M_0} \circ (I-L)(x_0), P_{M_0} \circ (I-L)(x))$$

which implies the result.

(3) $\Rightarrow$ (1). Let $x \in X$. If $y \in P_{M(x_0)}(x_0)$, then $y = z + Lx_0$ for some $z \in P_{M_0}(x_0 - Lx_0)$, and

$$\begin{aligned} d(y, P_{M(x)}(x)) &= d(z + Lx_0, P_{M_0}(x - Lx) + Lx) \\ &\leqslant d(z, P_{M_0}(x - Lx)) + \|Lx - Lx_0\| \\ &\leqslant h(P_{M_0}(x_0 - Lx_0), P_{M_0}(x - Lx)) + \|L\| \, \|x - x_0\|. \end{aligned}$$

Thus

$$h(P_{M(x_0)}(x_0), P_{M(x)}(x)) \leqslant h(P_{M_0}(x_0 - Lx_0), P_{M_0}(x - Lx)) + \|L\| \, \|x - x_0\|.$$

Since $P_{M_0}|_{X_0}$ is l.s.c. at $x_0 - Lx_0$, the right side can be made arbitrarily close to zero by choosing x close to x_0. This proves that $P_{M(\cdot)}(\cdot)$ is l.s.c. at x_0. ∎

It is well-known that the metric projection onto a finite-dimensional subspace is u.s.c. (see [10; Theorem 3.1, p. 386]). Using the equivalence of (1) and (3) in Theorem 2.8, we immediately obtain the following corollary.

COROLLARY 2.9. *The parameter map $P_{M(\cdot)}(\cdot)$ is upper semicontinuous on X.*

We conclude this section by showing that the existence of a selection for $P_{M(\cdot)}(\cdot)$ having certain continuity properties is equivalent to the existence of an analogous selection for the (restriction to X_0 of the) metric projection onto M_0.

Recall that a **selection** for the set-valued mapping $F: X \to 2^Y \backslash \{\varnothing\}$ is any function $f: X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.

THEOREM 2.10. *The following statements are equivalent.*

(1) *$P_{M(\cdot)}(\cdot)$ has a continuous (resp., linear, Lipschitz continuous) selection;*

(2) *$P_{M_0} \circ (I - L)$ has a continuous (resp., linear, Lipschitz continuous) selection;*

(3) *$P_{M_0}|_{X_0}$ has a continuous (resp., linear, Lipschitz continuous) selection.*

Proof. Using (2.4.2), it is obvious that f is a selection for $P_{M(\cdot)}(\cdot)$ iff $f - L$ is a selection for $P_{M_0} \circ (I - L)$. Moreover, L is linear, hence Lipschitz continuous. The equivalence of the three statements now follows easily. ∎

Various characterizations of which metric projections admit continuous or Lipschitz continuous selections can be found in [6]. Analogous characterizations for linear selections are in [5].

3. THE HILBERT SPACE CASE

In this section, we obtain stronger and more detailed results in the special case when X is a Hilbert space.

Our set-up for this section is the following. Let X be a Hilbert space, M an n-dimensional linear subspace, and let $\{\phi_1, \phi_2, ..., \phi_m\}$ be $m \leqslant n$ linearly independent functionals in X^*. For each $x \in X$, let

$$M(x) = \{ y \in M \mid \phi_i(y) = \phi_i(x) \qquad (i = 1, 2, ..., m)\}.$$

Letting $y_i \in X$ denote the "representer" of ϕ_i, we can rewrite $M(x)$ as

$$M(x) = \{ y \in M \mid \langle y, y_i \rangle = \langle x, y_i \rangle \qquad (i = 1, 2, ..., m)\}. \tag{3.0.1}$$

As before, we define

$$\begin{aligned} X_0 &:= \{ x \in X \mid \phi_i(x) = 0 \qquad (i = 1, 2, ..., m)\} \\ &= \{ x \in X \mid \langle x, y_i \rangle = 0 \qquad (i = 1, 2, ..., m)\} \\ &= (\operatorname{span}\{ y_1, y_2, ..., y_m\})^\perp \end{aligned} \tag{3.0.2}$$

and

$$M_0 := M \cap X_0. \tag{3.0.3}$$

Also, as above, we assume that there exists a linearly independent set $\{z_1, z_2, ..., z_m\}$ in M so that

$$\langle z_j, y_i \rangle = \delta_{ij} \qquad (i, j = 1, 2, ..., m), \tag{3.0.4}$$

and we define $L: X \to M$ by

$$Lx = \sum_{i=1}^{m} \langle x, y_i \rangle z_i, \qquad x \in X. \tag{3.0.5}$$

LEMMA 3.1. *Given $x \in X$, let $y_0 \in M(x)$. Then $y_0 = P_{M(x)}(x)$ if and only if $x - y_0 \in M_0^\perp$.*

This is a consequence of the well-known orthogonality characterization of the error when approximating by subspaces, along with the fact that $M(x) = M_0 + Lx$ is just the translate of a subspace.

To apply this lemma in practice, we need to recognize when an element is in $M(x)$, and when an element is in $M_0^\perp$. To this end, we show that if $\{z_{m+1}, z_{m+2}, ..., z_n\}$ is *any* basis of M_0, then $\{z_1, ..., z_m, z_{m+1}, ..., z_n\}$ is a basis for M.

To see this, note that Lemma 2.2 implies that M_0 is $(n-m)$-dimensional. Let $\{z_{m+1}, ..., z_n\}$ be a basis for M_0. In particular, $\{z_{m+1}, z_{m+2}, ..., z_n\} \subset M$ and

$$\langle z_i, y_j\rangle = 0 \qquad (i = m+1, m+2, ..., n; j = 1, 2, ..., m). \tag{3.1.1}$$

But by (3.0.4), $\langle z_i, y_j\rangle = \delta_{ij}$ for $i, j = 1, 2, ..., m$. Hence if $z \in M_0 \cap \operatorname{span}\{z_1, z_2, ..., z_m\}$, we see that $z = \sum_1^m \beta_k z_k$. Thus for $j = 1, 2, ..., m$, since $z \in M_0$, we have that

$$0 = \langle z, y_j\rangle = \sum_{k=1}^{m} \beta_k \langle z_k, y_j\rangle = \beta_j.$$

Thus $z = 0$. This proves that

$$M_0 \cap \operatorname{span}\{z_1, z_2, ..., z_m\} = \{0\}. \tag{3.1.2}$$

From this we deduce that $\{z_1, ..., z_m, z_{m+1}, ..., z_n\}$ is linearly independent. [For if $\sum_1^n \alpha_i z_i = 0$, then

$$\sum_1^m \alpha_i z_i = -\sum_{m+1}^{n} \alpha_i z_i \in M_0 \cap \operatorname{span}\{z_1, z_2, ..., z_n\} = \{0\}$$

and since $\{z_1, z_2, ..., z_m\}$ and $\{z_{m+1}, z_{m+2}, ..., z_n\}$ are each linear independent, it follows that $\alpha_i = 0$ for all $i = 1, 2, ..., n$.] Since $\{z_1, ..., z_m, z_{m+1}, ..., z_n\}$ is contained in M and M is n-dimensional, we see that $\{z_1, ..., z_m, z_{m+1}, ..., z_n\}$ is a *basis* for M.

THEOREM 3.2. *Let $\{y_1, y_2, ..., y_m\}$ be as in* (3.0.1) *and suppose $\{z_1, z_2, ..., z_m\}$ in M satisfies* (3.0.4). *Let $\{z_{m+1}, z_{m+2}, ..., z_n\}$ be any basis of M_0. Then for each $x \in X$,*

$$P_{M(x)}(x) = \sum_{m+1}^{n} \alpha_i z_i + Lx, \tag{3.2.1}$$

where $Lx = \sum_1^m \langle x, y_i\rangle z_i$ and the scalars $\{a_{m+1}, \alpha_{m+2}, ..., \alpha_n\}$ are the unique solution to the linear system

$$\sum_{i=m+1}^{n} \alpha_i \langle z_i, z_j\rangle = \langle x - Lx, z_j\rangle \qquad (j = m+1, m+2, ..., n). \tag{3.2.2}$$

Moreover, if $\{z_{m+1}, z_{m+2}, ..., z_n\}$ is an orthonormal basis for M_0, then

$$\alpha_j = \langle x - Lx, z_j\rangle \qquad (j = m+1, m+2, ..., n) \tag{3.2.2$'$}$$

and thus

$$P_{M(x)}(x) = \sum_{m+1}^{n} \langle x - Lx, z_j \rangle\, z_j + Lx. \tag{3.2.3}$$

Proof. Let $y_0 \in M(x)$. Since $M(x) = M_0 + Lx$ by Theorem 2.4 (1), it follows that

$$y_0 = \sum_{m+1}^{n} \alpha_i z_i + Lx \tag{3.2.4}$$

for some scalars α_i. By Lemma 3.1, $y_0 = P_{M(x)}(x)$ if and only if $x - y_0 \in M_0^{\perp}$. That is,

$$\langle x - y_0, z_j \rangle = 0 \qquad (j = m+1, m+2, ..., n).$$

It follows that

$$\sum_{i=m+1}^{n} \alpha_i \langle z_i, z_j \rangle = \langle x - Lx, z_j \rangle \qquad (j = m+1, m+2, ..., n). \tag{3.2.5}$$

Since $\{z_{m+1}, z_{m+2}, ..., z_n\}$ is linearly independent, the determinant $\det[\langle z_i, z_j \rangle]_{i,j=m+1}^{n}$ is not zero (see, e.g., [3; p. 178, Theorem 8.7.2]). This verifies the first statement of the theorem.

The second statement is an immediate consequence of the first. ∎

Corollary 3.3. *If $\{y_1, y_2, ..., y_m\}$ is an orthonormal set in M and $\{z_{m+1}, z_{m+2}, ..., z_n\}$ is an orthonormal basis for M_0, then for any $x \in X$,*

$$P_{M(x)}(x) = \sum_{i=m+1}^{n} \langle x - Lx, z_i \rangle\, z_i + Lx, \tag{3.3.1}$$

where

$$Lx = \sum_{1}^{m} \langle x, y_i \rangle\, y_i. \tag{3.3.2}$$

Proof. From the theorem,

$$P_{M(x)}(x) = \sum_{m+1}^{n} \langle x - Lx, z_j \rangle\, z_j + Lx, \tag{3.3.3}$$

where

$$Lx = \sum_{i=1}^{m} \langle x, y_i \rangle\, z_i. \tag{3.3.4}$$

But if we choose $z_i = y_i$ for $i = 1, 2, ..., m$, then $\{z_1, z_2, ..., z_m\}$ is in M and satisfies (3.0.4). Substituting y_i for z_i in (3.3.4), we obtain (3.3.1) and (3.3.2). ▮

Since, in a Hilbert space, the metric projection onto any closed subspace M_0 is just the (linear) orthogonal projection onto M_0, we obtain the following consequence of Theorem 2.4 (1).

COROLLARY 3.4. *Under the hypothesis of Theorem* 3.2, *the parameter mapping* $P_{M(\cdot)}(\cdot)$ *is linear.*

(This result also follows indirectly from the equivalence of (1) and (3) in Theorem 2.10.)

4. APPROXIMATION FROM INTERPOLATING SUBSPACES

In this section we restrict our attention to approximation from interpolating subspaces. Using the characterization theorem below (Theorem 4.5), we show that each $x \in X$ has a unique best approximation in $M(x)$. In fact, the best approximations are actually strongly unique (Corollary 4.8). Finally, using the strong uniqueness of best approximations, we prove that the parameter map associated with this problem is pointwise Lipschitz continuous on X (Theorem 4.9).

DEFINITION 4.1. [1]. An n-dimensional subspace M of a normed linear space X is called an **interpolating subspace** if, for each set of n linearly independent functionals $\{\phi_1, \phi_2, ..., \phi_n\} \subseteq \text{ext } B(X^*)$ and each set of n real scalars $c_1, c_2, ..., c_n$, there is a unique element $y \in M$ such that

$$\phi_i(y) = c_i \qquad \text{for} \quad i = 1, 2, ..., n.$$

Equivalently, M is an interpolating subspace of X if, whenever $\{\phi_1, \phi_2, ..., \phi_n\}$ is a set of n linearly independent functionals in ext $B(X^*)$, $y \in M$, and $\phi_i(y) = 0$ for all $i = 1, 2, ..., n$, then $y = 0$.

The notion of interpolating subspace was introduced by Ault, Deutsch, Morris, and Olson [1] as a generalization of the classical Haar subspace in $C[a, b]$. In $C_0(T)$, T a locally compact Hausdorff space, the interpolating subspaces are precisely the Haar subspaces [1]. However, interpolating subspaces are rare in general. Wulbert [11] observed that if X is a smooth normed linear space, then the best approximations in a Chebyshev subspace of X are not strongly unique. In [1] it was shown that if M is an interpolating subspace of a normed linear space X, then M is a Chebyshev subspace and best approximations are strongly unique. It thus follows that a smooth normed linear space does not contain any interpolating subspace. In the case where (T, μ) is a σ-finite positive measure space, then $L_1(T, \mu)$ contains an interpolating subspace of dimension $n > 1$ if and only if T is the union of at least n atoms; while $L_1(T, \mu)$ contains a one-dimensional interpolating subspace if and only if T contains an atom [1]. In particular, the space ℓ_1 contains interpolating subspaces of every dimension $n \geqslant 1$.

Let M be an n-dimensional interpolating subspace of X, and fix a set $\{\phi_1, \phi_2, ..., \phi_m\}$ of m linearly independent functionals in ext $B(X^*)$, where $1 \leqslant m \leqslant n$. As before, for each $x \in X$, let

$$M(x) := \{ y \in M \mid \phi_i(x) = \phi_i(y), \qquad i = 1, 2, ..., m\}.$$

Note that if $m = n$, then $M(x)$ is a singleton for each $x \in X$. We shall henceforth assume that $m < n$. Recall that

$$X_0 := \{x \in X \mid \phi_i(x) = 0 \qquad (i = 1, 2, ..., m)\}$$

and

$$M_0 := M \cap X_0 = \{x \in M \mid \phi_i(x) = 0 \qquad (i = 1, 2, ..., m)\}.$$

LEMMA 4.2. *There exist m elements $z_1, z_2, ..., z_m$ in M such that*

$$\phi_i(z_j) = \delta_{ij} \qquad (i, j = 1, 2, ..., m). \tag{4.2.1}$$

Proof. Since M is interpolating, for each $j = 1, 2, ..., m$, and scalars $c_1 = \delta_{1j}, c_2 = \delta_{2j}, ..., c_m = \delta_{mj}$, there exists a unique $z_j \in M$ so that $\phi_i(z_j) = c_i$ $(i = 1, 2, ..., m)$. That is, (4.2.1) holds. ∎

LEMMA 4.3. *M_0 is an $(n - m)$-dimensional interpolating, hence Chebyshev, subspace in X_0.*

Proof. By Lemma 2.2, M_0 is an $(n - m)$-dimensional subspace of X_0. It remains to show it is interpolating in X_0. Let $\{\psi_{m+1}, \psi_{m+2}, ..., \psi_n\}$ be linearly independent in ext $B(X_0^*)$, $y_0 \in M_0$, and $\psi_i(y_0) = 0$ for $i = m + 1$, $m + 2, ..., n$. We must show $y_0 = 0$. By [10; p. 168], each ψ_i can be extended to an element $\phi_i \in$ ext $B(X^*)$ $(i = m + 1, m + 2, ..., n)$.

Claim. $\{\phi_1, ..., \phi_m, \phi_{m+1}, ..., \phi_n\}$ is linearly independent (in ext $B(X^*)$).

To see this, let $\sum_1^n \alpha_i \phi_i = 0$. Then for all $y \in X_0$, $\phi_i(y) = 0$ for $i = 1$, $2, ..., m$, and

$$0 = \sum_1^n \alpha_i \phi_i(y) = \sum_{m+1}^n \alpha_i \phi_i(y) = \sum_{m+1}^n \alpha_i \psi_i(y).$$

Since $\{\psi_{m+1}, \psi_{m+2}, ..., \psi_n\}$ is linearly independent in X_0^*, it follows that $\alpha_i = 0$ for $i = m+1, m+2, ..., n$. Thus $\sum_1^m \alpha_i \phi_i = 0$. By Lemma 4.2, for each $j = 1, 2, ..., m$,

$$0 = \sum_1^m \alpha_i \phi_i(z_j) = \alpha_j.$$

Thus $\alpha_i = 0$ for $i = 1, 2, ..., m$. This proves the claim.

From the claim it follows that $\{\phi_1, \phi_2, ..., \phi_n\}$ is linearly independent in ext $B(X^*)$. Since $y_0 \in M_0$, $\phi_i(y_0) = 0$ for $i = 1, 2, ..., m$. But by assumption, $\psi_i(y_0) = 0$ for $i = m+1, m+2, ..., n$. Thus, $\phi_i(y_0) = 0$ for $i = m+1, m+2, ..., n$. Since $y_0 \in M$ and M is interpolating, $y_0 = 0$. ∎

In contrast to Lemma 4.3 the following example shows that, in general, M_0 is not interpolating in the *whole* space X.

EXAMPLE 4.4. Let $X = C[0, 1]$, $M = \mathbb{P}_2 = \text{span}\{1, t, t^2\}$ be the subspace of $C[0, 1]$ of all algebraic polynomials of degree at most 2, and define ϕ_1 and ϕ_2 on $C[0, 1]$ by $\phi_1(f) = f(0)$ and $\phi_2(f) = f(1)$ for all $f \in C[0, 1]$. Then

$$M_0 = M(0) = \{p \in \mathbb{P}_2 \mid p(0) = p(1) = 0\} = \text{span}(t^2 - t).$$

Clearly, M_0 is a one-dimensional subspace of $C[0, 1]$. The element $p(t) = t(t-1)$ belongs to M_0, and has two zeros in the interval $[0, 1]$. Thus M_0 is not a Haar subspace of $C[0, 1]$, and consequently, M_0 is not an interpolating subspace of $C[0, 1]$.

Let $\{y_1, y_2, ..., y_{n-m}\}$ be a basis for M_0 and choose $\{z_1, z_2, ..., z_m\}$ in M as in Lemma 4.2. Then $\{z_1, z_2, ..., z_m\}$ is clearly linearly independent so it can be augmented by elements $z_{m+1}, z_{m+2}, ..., z_n$ so that $\{z_1, z_2, ..., z_n\}$ is a basis for M. We define $L: X \to M$ by

$$Lx = \sum_{i=1}^m \phi_i(x)\, z_i.$$

Just as in Section 2, for each $x \in X$, $Lx \in M(x)$ and

$$P_{M(x)}(x) = [P_{M_0} \circ (I - L) + L](x).$$

For any set of $n - m + 1$ linearly independent functionals $\{\psi_1, \psi_2, ..., \psi_{n-m+1}\}$ in X^*, we define the determinants $\Delta_i = \Delta_i(\psi_1, \psi_2, ..., \psi_{n-m+1})$ by

$$\Delta_i := \begin{vmatrix} \psi_1(y_1) & \cdots & \psi_{i-1}(y_1) & \psi_{i+1}(y_1) & \cdots & \psi_{n-m+1}(y_1) \\ \psi_1(y_2) & \cdots & \psi_{i-1}(y_2) & \psi_{i+1}(y_2) & \cdots & \psi_{n-m+1}(y_2) \\ \cdots & & & & & \\ \psi_1(y_{n-m}) & \cdots & \psi_{i-1}(y_{n-m}) & \psi_{i+1}(y_{n-m}) & \cdots & \psi_{n-m+1}(y_{n-m}) \end{vmatrix}.$$

We also recall

$$\mathscr{E}(x) := \{\phi \in \operatorname{ext} B(X^*) \mid \phi(x) = \|x\|\}.$$

The following theorem characterizes best approximations to any $x \in X$ from $M(x)$.

THEOREM 4.5. *Let $x \in X$ and $y_0 \in M(x)$. Then the following statements are equivalent.*

(1) $y_0 \in P_{M(x)}(x)$;

(2) $y_0 - Lx \in P_{M_0}(x - Lx)$;

(3) $0 \in \operatorname{co}\{(x^*(y_1), x^*(y_2), ..., x^*(y_{n-m})) \mid x^* \in \mathscr{E}(x - y_0)\}$;

(4) *There exist $n-m+1$ linearly independent functionals $\psi_i \in \mathscr{E}(x - y_0)$ such that*

$$\operatorname{sgn}(\Delta_i) = (-1)^{i+1} \operatorname{sgn}(\Delta_1) \qquad (i = 1, 2, ..., n-m+1);$$

(5) *There exist $n-m+1$ linearly independent functionals $\psi_i \in \mathscr{E}(x - y_0)$ and $n-m+1$ nonzero scalars λ_i such that*

(a) $\sum_1^{n-m+1} \lambda_i \psi_i \in M_0^{\perp}$, *and*

(b) $\operatorname{sgn}[\lambda_1 \psi_1(x - y_0)] = \cdots = \operatorname{sgn}[\lambda_{n-m+1} \psi_{n-m+1}(x - y_0)]$.

Proof. The equivalence of (1) and (2) (resp., (1) and (3)) follows from Theorem 2.4 (1) (resp., Theorem 2.6). The equivalence of (1), (4), and (5) is a consequence of Lemma 4.3 and [1; Theorem 4.1]. ∎

Remark. In Theorem 4.5, we can replace the set $\mathscr{E}(x - y_0)$ by the (subset)

$$\{\phi \in \operatorname{ext} B(X_0^*) \mid \phi(x - y_0) = \|x - y_0\|\}.$$

This is a subset of $\mathscr{E}(x - y_0)$ since each $\phi \in \operatorname{ext} B(X_0^*)$ may be extended to a functional in $\operatorname{ext} B(X^*)$. But the restriction of a functional in $\operatorname{ext} B(X^*)$ to X_0 is not necessarily in $\operatorname{ext} B(X_0^*)$.

COROLLARY 4.6. *Each $x \in X$ has a unique best approximation in $M(x)$.*

Proof. This follows since M_0 is interpolating in X_0 by Lemma 4.3, hence is Chebyshev in X_0 by [1; Theorem 2.2], and Corollary 2.5 (2). ∎

Actually, the best approximation to x from $M(x)$ is "strongly unique" in the sense described below.

THEOREM 4.7 [1, Theorem 6.1]. *Let Y be an interpolating subspace of X. Then for each $x \in X$, there exists a scalar $\gamma = \gamma(x) \in (0, 1]$ such that*

$$\|x - y\| \geqslant \|x - P_Y(x)\| + \gamma \, \|P_Y(x) - y\| \tag{4.7.1}$$

for all $y \in Y$.

COROLLARY 4.8 (Strong Uniqueness of Best Approximations). *For each $x \in X$, there exists $\theta = \theta(x) \in (0, 1]$ such that*

$$\|x - y\| \geqslant \|x - P_{M(x)}(x)\| + \theta \, \|P_{M(x)}(x) - y\| \tag{4.8.1}$$

for all $y \in M(x)$.

Proof. By Lemma 4.3, M_0 is an interpolating subspace in X_0. Since $x - Lx \in X_0$ for each $x \in X$, and $y - Lx \in M_0$ for each $y \in M(x)$, it follows by Theorem 4.7 that there exists $\theta = \theta(x) \in (0, 1]$ so that

$$\begin{aligned} \|x - Lx - (y - Lx)\| \geqslant \|x - Lx - P_{M_0}(x - Lx)\| \\ + \theta \, \|P_{M_0}(x - Lx) - (y - Lx)\|. \end{aligned}$$

for each $y \in M(x)$. Using (2.4.2), this is equivalent to (4.8.1). ∎

Finally, we show that the parameter map is *pointwise Lipschitz continuous.* Recall that a mapping f from one normed linear space X into another Y is said to be **pointwise Lipschitz continuous at** $x \in X$ if there is a constant $\lambda = \lambda(x) > 0$ such that

$$\|f(x) - f(y)\| \leqslant \lambda \, \|x - y\| \qquad \text{for all} \quad y \in X.$$

It is well-known, and easy to prove (see [2; p. 82, Freud's theorem]) that strong uniqueness of best approximations from a Chebyshev subspace implies the pointwise Lipschitz continuity of its metric projection at each point. By Lemma 4.3, M_0 is an interpolating subspace in X_0. Hence, by Theorem 4.7, best approximations from M_0 to each $x \in X_0$ are strongly unique. It follows that for each $x \in X_0$, there exists $\lambda = \lambda(x) > 0$ so that

$$\|P_{M_0}(x) - P_{M_0}(y)\| \leqslant \lambda \, \|x - y\| \tag{4.8.2}$$

for all $y \in X_0$. By Lemma 2.3, $x - Lx$ is in X_0 for each $x \in X$. Thus for each $x \in X$ and $\lambda' := \lambda'(x) := \lambda(x - Lx)$, we deduce

$$\|P_{M_0}(x - Lx) - P_{M_0}(y - Ly)\| \leqslant \lambda' \|x - Lx - (y - Ly)\| \tag{4.8.3}$$

for each $y \in X$. By Theorem 2.4 (2), we see that

$$\begin{aligned} \|P_{M(x)}(x) - P_{M(y)}(y)\| &= \|P_{M_0}(x - Lx) + Lx - [P_{M_0}(y - Ly) + Ly]\| \\ &\leqslant \|P_{M_0}(x - Lx) - P_{M_0}(y - Ly)\| + \|Lx - Ly\| \\ &\leqslant \lambda' \|x - Lx - (y - Ly)\| + \|Lx - Ly\| \\ &\leqslant \lambda' \|x - y\| + (\lambda' + 1) \|Lx - Ly\| \\ &\leqslant \mu(x) \|x - y\|, \end{aligned}$$

where $\mu(x) = \lambda'(x) + (\lambda'(x) + 1) \|L\|$.

Thus we have proved the following.

THEOREM 4.9. *The parameter map $P_{M(\cdot)}(\cdot)$ is pointwise Lipschitz continuous on X. That is, for each $x \in X$, there exists a constant $\mu(x) > 0$ such that*

$$\|P_{M(x)}(x) - P_{M(y)}(y)\| \leqslant \mu(x) \|x - y\| \tag{4.9.1}$$

for all $y \in X$.

In the special case when M is a Haar subspace in $X = C[a, b]$, it was proved in [4] that (4.9.1) holds for those $y \in C[a, b]$ which satisfy the additional restriction: $M(y) = M(x)$. Thus Theorem 4.9 shows that this additional restriction in [4] may be omitted.

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